Int. J. Solids Structures Vol. 25, No. 5, pp. 515–526, 1989
Printed in Great Britain.

0020-7683/89 $3.00+ .00

REMARKS ON A DYNAMICAL HIGHER-ORDER THEORY OF LAMINATED PLATES AND ITS APPLICATION IN RANDOM VIBRATION RESPONSE

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(*Received* 16 *December* 1987; *in revised form* 2 *October* 1988)

Abstract—This paper presents an analysis of the equations governing the dynamics of shear-deformable composite plates, without recourse to a variational procedure. It is noted that the operator associated with the governing equations is nonsymmetric; using a first-order perturbation technique, it is shown to be positive-definite. In addition, using the biorthogonality condition, the dynamic response of the plate is formulated.

1. INTRODUCTION

Substantiation of shear-deformation theories of composite plates and shells has been the object of increasing attention during the last years. A review of composite structures is given by Bert and Francis (1974), and a critical review of transverse shear-deformable plate theories by Librescu and Reddy (1987).

In this paper we will refer to the basic approaches, namely, (i) a higher-order shear deformation theory (HSDT), derived on the basis of representation of the displacement field as per eqn (1), without recourse to a variational principle (Librescu, 1968; Librescu and Reddy, 1987), and (ii) theories derived through a variational principle and based on the above representation (Reddy and Phan, 1985), or on a linear representation (FSDT) of the displacement field through the plate thickness (Yang *et al.*, 1965; Whitney and Pagano, 1970). These two basic theories will be referred to later as A and B, respectively.

Although the matrix associated with A is not self-adjoint, it is expected, considering the conservative character of the problem, to be positive-definite, and will be shown to be so using a first-order perturbation technique.

In the numerical examples, the response of a rectangular cross-ply laminated plate excited by a stationary random load is considered, and the results are compared with their counterparts in B.

2. REFINED HIGHER-ORDER THEORY

In the subsequent analysis, the distribution of the displacement field across the plate thickness is considered as in Cederbaum *et al.* (1987):

$$U_1 = z\psi_x - \frac{4}{3h^2} z^3(\psi_x + W_{,x})$$

$$U_2 = z\psi_y - \frac{4}{3h^2} z^3(\psi_y + W_{,y})$$

$$U_3 = W \tag{1}$$

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where U_1, U_2 and U_3 are the components of the 3D displacement vector in the x-, y- and z-directions, respectively; ψ_x and ψ_y denote the rotation of the normals to the mid-plane about the y- and x-axes, respectively, while $(\ )_{,j}$ denotes the partial derivative with respect to the indicated coordinate.

The above representation of the displacement field yields a parabolic distribution of transverse shear strains across the plate thickness and the condition of zero in-plane loads on the bounding planes of the plate (see Fig. 1). By this means, the need to introduce a transverse shear correction factor, as in the case of FSDT, is obviated.

For an orthotropic material, in which the elastic axes of the layer coincide with the geometrical ones, the pertinent constitutive equations may be expressed as

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & & & \\ Q_{12} & Q_{22} & & 0 & \\ & & Q_{44} & & \\ & 0 & & Q_{55} & \\ & & & & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} + \sigma_3 \begin{Bmatrix} R_{11} \\ R_{22} \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{2}$$

where

$$Q_{11} = \frac{E_1}{\Omega}; \quad Q_{22} = \frac{E_2}{\Omega}; \quad Q_{12} = \frac{\nu_{12}E_2}{\Omega}; \quad \Omega \equiv 1 - \nu_{12}\nu_{21}$$

$$Q_{44} = G_{23}; \quad Q_{55} = G_{13}; \quad G_{66} = G_{12}$$

and

$$R_{11} = \frac{E_1}{E_3}\frac{\nu_{31} + \nu_{21}\nu_{32}}{\Omega}; \quad R_{22} = \frac{E_2}{E_3}\frac{\nu_{32} + \nu_{12}\nu_{31}}{\Omega}$$

are the reduced elastic constants.

The distribution of the transverse normal stress, σ_3, can be obtained by integration across the segment $[0, z)$, of the equation of motion of the 3D elasticity theory, written in the absence of body forces as

$$\sigma_{i3,i} = \rho \ddot{U}_3 \quad i = 1, 2, 3 \tag{3}$$

where ρ is mass density and the dots denote the time derivatives. This yields

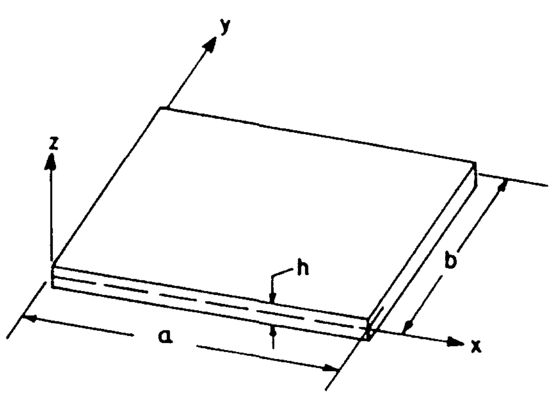

Fig. 1. Geometry and coordinate system for a rectangular plate.

$$\sigma_3 = z[\rho\ddot{U}_3 - Q_{55}(\psi_{x,x} + W_{,xx}) - Q_{44}(\psi_{y,y} + W_{,yy})] - \frac{4}{3H^2} z^3 (Q_{55} W_{,xx} + Q_{44} W_{,yy}). \quad (4)$$

The stress resultants L_{ij} and stress couples L_{i3}, $(i, j = 1, 2)$, involved in the bending equations of motion of plate theory are defined as

$$L_{ij} = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} z\sigma_{ij}^{(k)}\,\mathrm{d}z$$

$$L_{i3} = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} z\sigma_{i3}^{(k)}\,\mathrm{d}z \quad (5)$$

where N denotes the total number of layers.

The equations of motion necessary for solution of this problem as expressed in terms of the 2D quantities in eqn (5), are

$$L_{1j,j} - L_{13} = f_1$$

$$L_{2i,i} - L_{23} = f_2 \quad i, j = 1, 2$$

$$L_{i3,i} + P_3 = f_3 \quad (6)$$

where P_3 denotes the transverse external load, while

$$f_i = \int_{-h/2}^{h/2} \rho z \ddot{U}_i\,\mathrm{d}z$$

and

$$f_3 = \int_{-h/2}^{h/2} \rho z \ddot{U}_3\,\mathrm{d}z$$

are the rotatory and transversal inertia terms, respectively.

Equations (6) may be obtained through integration of the equations of motion of the 3D elasticity theory across the plate thickness. Finally, the governing equations associated with bending theory are obtained by expressing the stress resultants and stress couples in eqns (6) in terms of the unknowns ψ_x, ψ_y and W. Using in addition the proportional damping model (C being the damping factor), the governing equations read:

$$\left(D_{11} - \frac{4}{3h^2}F_{11}\right)\psi_{x,xx} + \left(D_{66} - \frac{4}{3h^2}F_{66}\right)\psi_{x,yy} + \left(D_{12} - \frac{4}{3h^2}F_{12}\right)\psi_{y,xy}$$

$$+ \left(D_{66} - \frac{4}{3h^2}F_{66}\right)\psi_{y,xy} - \frac{4}{3h^2}F_{11}W_{,xxx} - \frac{4}{3h^2}(F_{12} + F_{66})W_{,xyy}$$

$$- \left(A_{55} - \frac{4}{h^2}D_{55}\right)(\psi_x + W_{,x}) - \left(\bar{D}_{55} - \frac{4}{3h^2}\bar{F}_{55}\right)(W_{,xxx} + \psi_{x,xx})$$

$$- \left(\bar{D}_{44} - \frac{4}{3h^2}\bar{F}_{44}\right)(W_{,xyy} + \psi_{y,xy})$$

$$= \left(I_3 - \frac{4}{3h^2}I_5\right)(\ddot{\psi}_x + C\dot{\psi}_x) - \left(\frac{4}{3h^2}I_5 + \bar{I}_{31}\right)(\ddot{W}_{,x} + C\dot{W}_{,x}),$$

$$\left(D_{22}-\frac{4}{3h^2}F_{22}\right)\psi_{y,yy}+\left(D_{66}-\frac{4}{3h^2}F_{66}\right)\psi_{y,xx}+\left(D_{21}-\frac{4}{3h^2}F_{21}\right)\psi_{x,xy}$$

$$+\left(D_{66}-\frac{4}{3h^2}F_{66}\right)\psi_{x,xy}-\frac{4}{3h^2}F_{22}W_{,yyy}-\frac{4}{3h^2}(F_{21}+F_{66})W_{,yxx}$$

$$-\left(A_{44}-\frac{4}{h^2}D_{44}\right)(\psi_y+W_{,y})-\left(\bar{D}_{44}-\frac{4}{3h^2}\bar{F}_{44}\right)(W_{,yyy}+\psi_{y,yy})$$

$$-\left(\bar{D}_{55}-\frac{4}{3h^2}\bar{F}_{55}\right)(W_{,xxx}+\psi_{x,xx})$$

$$=\left(I_3-\frac{4}{3h^2}I_5\right)(\ddot{\psi}_y+C\dot{\psi}_y)-\left(\frac{4}{3h^2}I_5+\bar{I}_{32}\right)(\ddot{W}_{,y}+C\dot{W}_{,y}),$$

$$\left(A_{55}-\frac{4}{h^2}D_{55}\right)(W_{,xx}+{}_{x,x})+\left(A_{44}-\frac{4}{h^2}D_{44}\right)(W_{,yy}+\psi_{y,y})+P_z=I_1(\ddot{W}+C\dot{W}) \qquad (7)$$

with the rigidities and inertia terms defined as:

$$(A_{ij},D_{ij},F_{ij})=\int_{-h/2}^{h/2}Q_{ij}(1,z^2,z^4)\,dz \qquad (i,j=1,2,6)$$

$$(A_{ij},D_{ij})=\int_{-h/2}^{h/2}Q_{ij}(1,z^2)\,dz \qquad (i,j=4,5)$$

$$(\bar{D}_{44},\bar{F}_{44})=\int_{-h/2}^{h/2}Q_{44}R_{11}(z^2,z^4)\,dz$$

$$(\bar{D}_{55},\bar{F}_{55})=\int_{-h/2}^{h/2}Q_{55}R_{22}(z^2,z^4)\,dz$$

$$(I_1,I_3,I_5)=\int_{-h/2}^{h/2}\rho(1,z^2,z^4)\,dz$$

$$\bar{I}_{3i}=\int_{-h/2}^{h/2}\rho R_{ii}z^2\,dz \qquad (i=1,2).$$

3. DYNAMIC RESPONSE

Equations (7) constitute a set of partial differential equations of the sixth order. For the case of a simply-supported rectangular panel ($a\times b$), the boundary conditions read

$$W=\psi_y=L_{11}=0 \quad \text{at } x=0,a$$
$$W=\psi_x=L_{22}=0 \quad \text{at } y=0,b. \qquad (8)$$

The solution functions are then represented in a form that satisfies exactly the boundary conditions

$$\psi_x(x,y,t)=\sum_{m,n}\hat{X}_{mn}\cos\alpha x\sin\beta y\,T_{mn}(t)\equiv\sum_{m,n}X_{mn}T_{mn}$$

$$\psi_y(x,y,t)=\sum_{m,n}\hat{Y}_{mn}\sin\alpha x\cos\beta y\,T_{mn}(t)\equiv\sum_{m,n}Y_{mn}T_{mn}$$

$$W(x,y,t)=\sum_{m,n}\hat{W}_{mn}\sin\alpha x\sin\beta y\,T_{mn}(t)\equiv\sum_{m,n}W_{mn}T_{mn} \qquad (9)$$

where $\alpha = m\pi/a$; $\beta = n\pi/b$; $\hat{X}_{mn}$, $\hat{Y}_{mn}$, $\hat{W}_{mn}$ are the coefficients of the natural mode shapes associated with the free vibration problem while $T_{mn}(t)$ denote the generalized coordinates. The transverse loading function is given by

$$P_3 = P_3(x, y, t) = \sum_{m,n} q_{mn} \sin \alpha x \sin \beta y \, F_{mn}(t) \tag{10}$$

where q_{mn} are the Fourier coefficients. For the free vibration problem $F_{mn}(t) \equiv 0$, $C \equiv 0$ and $T_{mn}(t) = e^{i\omega_{mn}t}$ $(i = \sqrt{-1})$; using (9) in the governing equations (7), we obtain the eigenvalue problem in the form

$$[[K]_{mn} - \omega_{mn}^2 [M]] \{\Delta\}_{mn} = \{0\} \tag{11}$$

where

$$\{\Delta\}_{mn}^T = \{\hat{X}_{mn}, \hat{Y}_{mn}, \hat{W}_{mn}\}.$$

Both $[K]$ and $[M]$ are real and nonsymmetric matrices, and since $[M]$ is also nonsingular, we can multiply eqn (11) by $[M]^{-1}$ from the left to obtain for each mn

$$[M]^{-1}[K]\{\Delta\} = \omega^2 [M]^{-1}[M]\{\Delta\} = \omega^2 [I]\{\Delta\} \tag{12}$$

and by writing $[A] = [M]^{-1}[K]$, the eigenvalue problem is obtained in the form

$$[A]\{\Delta\} = \omega^2 [I]\{\Delta\} \tag{13}$$

where $[A]$ is likewise real and nonsymmetric.

Consider now the eigenvalue problem associated with the adjoint operator $[A]^T$. Its eigenvalues $\bar{\omega}^2$ are the same as those of $[A]$, so we can write

$$[A]^T\{\bar{\Delta}\} = \bar{\omega}^2 [I]\{\bar{\Delta}\} = \omega^2 [I]\{\bar{\Delta}\}. \tag{14}$$

For this case the biorthogonality condition (Meirovitch, 1980) is applied

$$(\omega_{mn}^2 - \bar{\omega}_{pq}^2) \int_0^a \int_0^b \{\Delta\}_{mn} \{\bar{\Delta}\}_{pq} \, dy \, dx = 0 \tag{15}$$

where the barred quantities are associated with the eigenvalue problem of $[A]^T$.

Using the modal analysis technique, the decoupled differential equation for $T_{mn}(t)$ is

$$\ddot{T}_{mn}(t) + C\dot{T}_{mn}(t) + \omega_{mn}^2 T_{mn}(t) = \frac{1}{J_{mn}} F_{mn}(t) \tag{16}$$

where

$$C = 2\xi_{mn}\omega_{mn} \qquad F_{mn}(t) = \int_0^a \int_0^b \bar{W}_{mn} P_3(x, y, t) \, dy \, dx$$

while J_{mn} stands for the norm (generalized mass), defined as

$$J_{mn,pq} = \begin{cases} J_{mn,mn} = J_{mn} & m = p, n = q \\ 0 & \text{otherwise} \end{cases}$$

where

$$J_{mn} = \int_0^a \int_0^b \Big\{ I_1 W_{mn} \bar{W}_{mn} + I_3 (X_{mn} \bar{X}_{mn} + Y_{mn} \bar{Y}_{mn})$$

$$- \frac{8}{3h^2} I_5 (X_{mn} \bar{X}_{mn} + Y_{mn} \bar{Y}_{mn} + X_{mn} \bar{W}_{mn,x} + Y_{mn} \bar{W}_{mn,y})$$

$$- \frac{16}{9h^4} I_7 [X_{mn} \bar{X}_{mn} + Y_{mn} \bar{Y}_{mn} + 2(X_{mn} \bar{W}_{mn,x} + Y_{mn} \bar{W}_{mn,y})$$

$$+ W_{mn,x} \bar{W}_{mn,x} + W_{mn,y} \bar{W}_{mn,y}] \Big\} \, \mathrm{d}y \, \mathrm{d}x. \tag{17}$$

Under homogeneous initial conditions, the solution of eqn (16) is

$$T_{mn}(t) = \frac{1}{J_{mn}} \int_0^t F_{mn}(\tau) h_{mn}(t-\tau) \, \mathrm{d}\tau \tag{18}$$

then with eqn (9), and following Elishakoff (1983), the transverse displacement is expressed as

$$W(x, y, t) = \sum_{m,n} \frac{1}{J_{mn}} \int_{-\infty}^{\infty} F_{mn}(\tau) h_{mn}(t-\tau) \, \mathrm{d}\tau$$

$$= \sum_{m,n} W_{mn} \frac{1}{J_{mn}} \int_{-\infty}^{\infty} \bar{F}_{mn}(\omega) H_{mn}(\omega) e^{i\omega t} \, \mathrm{d}\omega \tag{19}$$

where

$$\bar{F}_{mn}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F_{mn}(t) e^{-i\omega t} \, \mathrm{d}t$$

while $H_{mn}(\omega)$ is the complex frequency response function associated with the mn mode,

$$H_{mn}(\omega) = 1/(\omega_{mn}^2 - \omega^2 + 2i\xi_{mn}\omega_{mn}\omega) \equiv 1/L_{mn}(\omega).$$

4. RANDOM VIBRATION ANALYSIS

For stationary excitation with zero mean, the cross-correlation of the transverse displacement function is expressed as

$$R_W(x_1, y_1, x_2, y_2, \tau) = \sum_{m,n} \sum_{p,q} W_{mn}(x_1, y_1) W_{pq}(x_2, y_2) \int_{-\infty}^{\infty} S_{Q_{mn}Q_{pq}}(\omega) H_{mn}(\omega) H_{pq}^*(\omega) e^{i\omega\tau} \, \mathrm{d}\omega \tag{20}$$

where

$$S_{Q_{mn}Q_{pq}}(\omega) = \frac{1}{J_{mn} J_{pq}} \int_0^a \int_0^b \int_0^a \int_0^b S_F(x_1, y_1, x_2, y_2, \omega) \times W_{mn}(x_1, y_1) W_{pq}(x_2, y_2) \, \mathrm{d}y_2 \, \mathrm{d}x_2 \, \mathrm{d}y_1 \, \mathrm{d}x_1) \tag{21}$$

while

$$S_F(x_1, y_1; x_2, y_2, \omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_F(x_1, y_1; x_2 y_2, \tau)e^{-i\omega\tau}\,d\tau$$

denotes the cross-spectral density function of the applied load.

If the plate is driven by a point load at $(\bar{x}, \bar{y})$, random in time, and characterized by an ideal white-noise correlation function, we have

$$R_F(x_1, x_2, y_1, y_2, \tau) = R\delta(x_1-\bar{x})\delta(x_2-\bar{x})\delta(y_1-\bar{y})\delta(y_2-\bar{y})\delta(\tau) \tag{22}$$

and

$$S_F(x_1, x_2, y_1, y_2, \omega) = \frac{R}{2\pi}\delta(x_1-\bar{x})\delta(x_2-\bar{x})\delta(y_1-\bar{y})\delta(y_2-\bar{y}) \tag{23}$$

$$S_{Q_{mn}Q_{pq}}(\omega) = \frac{1}{J_{mn}J_{pq}}\int_{A_1}\int_{A_2}\left[\frac{R}{2\pi}\delta(x_1-\bar{x})\delta(x_2-\bar{x})\delta(y_1-\bar{y})\delta(y_2-\bar{y})\right.$$
$$\left.\sin\frac{m\pi}{a}x_1\sin\frac{n\pi}{b}y_1\sin\frac{p\pi}{a}x_2\sin\frac{q\pi}{b}y_2\right]dA_2\,dA_1$$
$$= \frac{1}{J_{mn}J_{pq}}S_0\sin\frac{m\pi}{a}\bar{x}\sin\frac{n\pi}{b}\bar{y}\sin\frac{p\pi}{a}\bar{x}\sin\frac{q\pi}{b}\bar{y} \tag{24}$$

where $S_0 = R/2\pi$ and $dA_i = dx_i\,dy_i$, $(i = 1, 2)$.

For the case where the load is applied at the center of the plate, i.e. $\bar{x} = a/2$; $\bar{y} = b/2$, we have

$$S_{Q_{mn}Q_{pq}}(\omega) = \frac{1}{J_{mn}J_{pq}}S_0\left[(-1)^{(m-1)/2}\quad(-1)^{(n-1)/2}\quad(-1)^{(p-1)/2}\quad(-1)^{(q-1)/2}\right]$$
$$m,n,p,q = 1, 3, 5\ldots \tag{25}$$

and the mean-square of the displacement function at the driven point is

$$R_W\left(\frac{a}{2}, \frac{b}{2}, \frac{a}{2}, \frac{b}{2}, 0\right) = \sum_{m,n}\sum_{p,q} W_{mn}\left(\frac{a}{2}, \frac{b}{2}\right)W_{pq}\left(\frac{a}{2}, \frac{b}{2}\right)\int_{-\infty}^{\infty}\frac{S_{Q_{mn}Q_{pq}}(\omega)}{L_{mn}(\omega)L^*_{pq}(\omega)}\,d\omega. \tag{26}$$

The natural frequencies were found to be well separated (Cederbaum *et al.*, 1987) and for the case of light damping, the autocorrelation terms only are taken into account, so that

$$R_W\left(\frac{a}{2}, \frac{b}{2}, \frac{a}{2}, \frac{b}{2}, 0\right) \simeq S_0\sum_{m,n}\frac{1}{J^2_{mn}}\sin^2\frac{m\pi}{2}\sin^2\frac{n\pi}{2}\int_{-\infty}^{\infty}\frac{d\omega}{|L_{mn}(\omega)|^2}$$
$$= S_0\sum_{m,n}\frac{1}{J^2_{mn}}\frac{\pi}{2\xi_{mn}\omega^3_{mn}} \equiv E[W^2] \quad m, n = 1, 3, 5, \ldots \tag{27}$$

5. FIRST-ORDER PERTURBATION OF THE EIGENVALUE PROBLEM

Let A_o be $n\times n$ real matrix, whose eigenvalues λ_{oi} $i = 1, 2, \ldots, n$ are distinct and whose right and left eigenvectors $\{\Delta_o\}$ and $\{\bar{\Delta}_o\}$, respectively, are normalized to satisfy

$$\{\bar{\Delta}_o\}^T_j\{\Delta_o\}_i = \delta_{ij} \quad\text{and}\quad \{\bar{\Delta}_o\}^T_j A_o\{\Delta_o\}_i = \lambda_{oi}\delta_{ij}. \tag{28}$$

Consider also the $n\times n$ matrix δA_o, representing a small variation of A_o, whose eigenvalues and eigenvectors are $\delta\lambda_{oi}$ and $\{\delta\Delta_o\}_i$ $\{\delta\bar{\Delta}_o\}_i$, respectively.

For the perturbed matrix A, defined as

$$A = A_o + \delta A_o \tag{29}$$

the assumption of a first order perturbation implies

$$\lambda_i = \lambda_{oi} + \delta\lambda_{oi}$$
$$\{\Delta\}_i = \{\Delta_o\}_i + \{\delta\Delta_o\}_i; \quad \{\bar{\Delta}\}_i = \{\bar{\Delta}_o\}_i + \{\delta\bar{\Delta}_o\}_i \tag{30}$$

which enables us to determine $\delta\Delta_{oi}$, $\{\delta\Delta_o\}_i$ and $\{\delta\bar{\Delta}_o\}_i$ from the already known A_o, δA_o, λ_{oi}, $\{\Delta_o\}_i$ and $\{\bar{\Delta}_o\}_i$. This leads to the following expressions, Meirovitch (1980):

$$\delta\lambda_{oi} = \{\bar{\Delta}_o\}_i^T \delta A_o \{\Delta_o\}_i$$
$$\{\delta\Delta_o\}_i = \sum_{k=1}^{n} \frac{\{\bar{\Delta}_o\}_k^T \delta A_o \{\Delta_o\}_i}{\lambda_{oi} - \lambda_{ok}} \{\Delta_o\}_k \quad i, k = 1, 2, \ldots, n; \quad i \neq k$$
$$\{\delta\bar{\Delta}_o\}_j = \sum_{k=1}^{n} \frac{\{\Delta_o\}_k^T \delta A_o \{\bar{\Delta}_o\}_j}{\lambda_{oj} - \lambda_{ok}} \{\bar{\Delta}_o\}_k \quad j, k = 1, 2, \ldots, n; \quad j \neq k. \tag{31}$$

For the case where A_o is symmetric, so that $\{\Delta_o\}_i = \{\bar{\Delta}_o\}_i$, and δA_o is real, we obtain

$$\delta\lambda_{oi} = \{\Delta_o\}_i \delta A_o \{\Delta_o\}_i \tag{32}$$

which is a real number, and

$$\{\delta\Delta_o\}_i = \{\delta\bar{\Delta}_o\}_i \tag{33}$$

which implies that λ_i is a real number as well, and

$$\{\Delta\}_i = \{\Delta_o\}_i + \{\delta\Delta_o\}_i = \{\bar{\Delta}_o\}_i + \{\delta\bar{\Delta}_o\}_i = \{\bar{\Delta}\}_i. \tag{34}$$

From eqn (34) we conclude that although the perturbed matrix A might be non-symmetric, its right and left eigenvectors coincide within the first order perturbation, and the norm associated with A may be computed by the right eigenvectors only.

Next, we would like to show that for the case where δA_o is nonsymmetric, A, although not self-adjoint, may be positive–definite. For this purpose we use the following theorem, based on the fact that the eigenvalues of A depend continuously on its coefficients (Franklin, 1969):

"Let $\mu_1 \ldots \mu_s$ be the different eigenvalues of an $n \times n$ matrix $A = (a_{ij})$. Let the eigenvalue μ_j have multiplicity m_j, where $\Sigma m_j = n$. Then, for all sufficient small $\varepsilon > 0$ there is a number $\gamma = \gamma(\varepsilon) > 0$ such that if $|b_{ij} - a_{ij}| \leqslant \gamma$ for $i, j = 1, 2, \ldots, n$ then the matrix $B = (b_{ij})$ has exactly m_j eigenvalues in the circle $|\lambda - \mu_j| < \varepsilon$ for each $j = 1, 2, \ldots, s$."

This theorem implies, for our case, that once λ_{o1} (the smallest eigenvalue of A_o) is sufficiently positive (removed from zero), and δA_o sufficiently small, then, even if all the eigenvalues of the perturbation matrix, $\delta\lambda_{oi}$, are negative (they were shown to be real)—those of A, λ_i, are positive, allowing us to conclude that A is positive–definite.

We now would like to apply the previous HSDT derivation of the non self-adjoint system for determination of its eigenvalues and eigenvectors. To this end, let A_o be the matrix obtained by the other HSDT version, (category B) and A its counterpart obtained by the present HSDT version. All that has to be shown is that δA_o is small compared with A_o. This will be illustrated in the following numerical example.

Consider a rectangular, symmetric, cross-ply laminate, composed of four layers (0°, 90°, 90°, 0°) of equal thickness. The material of each layer, consisting of a woven graphite fabric and carbon matrix, has the following engineering constants, given by Pagano and Soni (1986)

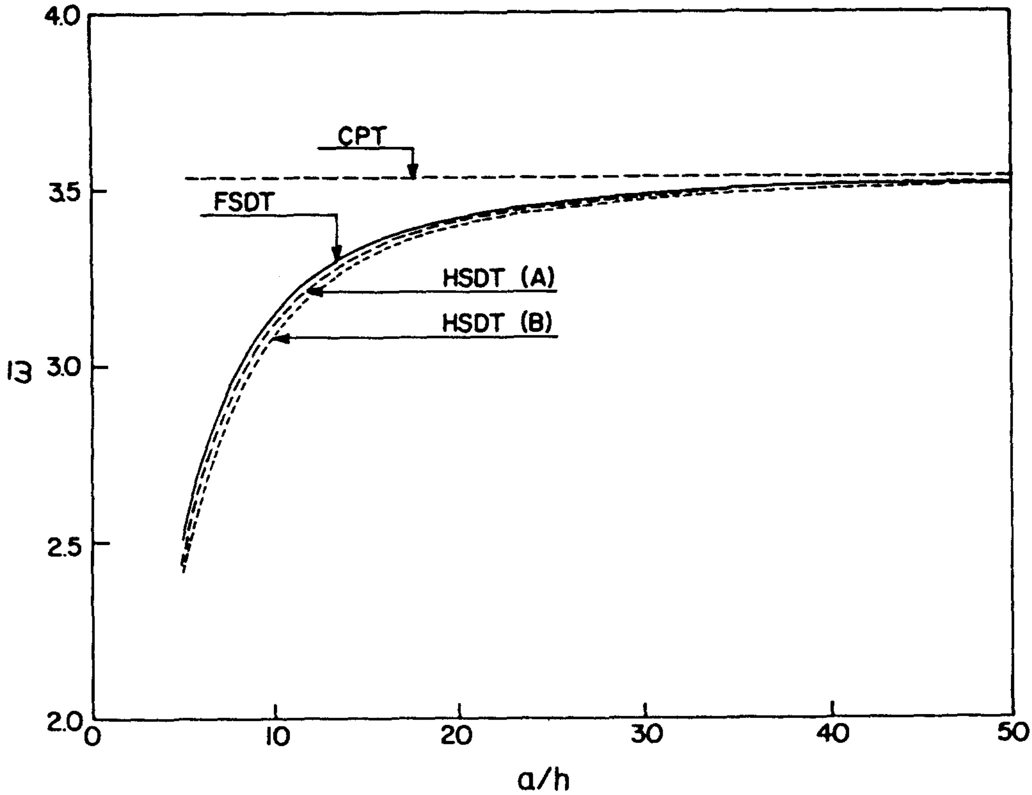

Fig. 2. Normalized fundamental frequencies vs a/h $[\bar{\omega} = \omega(a^2/h)(\rho/E_2)^{1/2}]$.

$$E_1 = 25.1\,\text{MSI} \quad G_{12} = 1.36\,\text{MSI} \quad \nu_{12} = 0.031$$
$$E_2 = 4.8\,\text{MSI} \quad G_{13} = 1.2\,\text{MSI} \quad \nu_{13} = 0.25 \quad \text{and} \quad \rho = 0.075\,\text{PCI}.$$
$$E_3 = 0.75\,\text{MSI} \quad G_{23} = 0.47\,\text{MSI} \quad \nu_{23} = 0.171$$

Note: The displacement field of eqn (1) is also used in the HSDT (B) version, where the right and left eigenvectors coincide, and the norm is:

$$J_{mn} = \int_0^a \int_0^b \Big\{ I_1 W_{mn}^2 + I_3(X_{mn}^2 + Y_{mn}^2) - \frac{8}{3h^2} I_5(X_{mn}^2 + Y_{mn}^2 + X_{mn} W_{mn,x} + Y_{mn} W_{mn,y}) - \frac{16}{9h^4} I_7[X_{mn}^2 + Y_{mn}^2 + 2(X_{mn} W_{mn,x} + Y_{mn} W_{mn,y}) + W_{mn,x}^2 + W_{mn,y}^2] \Big\}\, \mathrm{d}y\, \mathrm{d}x. \quad (35)$$

Table 1 shows the A, A_o, δA_o and $\delta A_o/A_o$ matrices, for the first mode, with $a = b = 10\,h$. In general, A differs from A_o by less than 10%, which is of the same order as in the example of Ryland and Meirovitch (1980). (The difference is even smaller for $a/h > 10$.) Figure 2

Table 1. Matrices A (HSDT (A)), A_o (HSDT(B)), δA_o and $\delta A_o/A_o$ for m, n = 1, 1 and a/h = 10.

	A_o			A	
0.561E+11	0.656E+10	0.112E+11	0.566E+11	0.753E+10	0.105E+11
0.895E+13	0.433E+13	0.862E+11	0.102E+14	0.478E+13	0.793E+11
0.165E+14	0.827E+11	0.584E+13	0.145E+14	0.739E+11	0.547E+13
	$\delta A_o = A - A_o$			$\delta A_o/A_o$	
0.501E+09	0.966E+09	−0.751E+09	0.893E−02	0.147E+00	0.669E−01
0.123E+13	0.448E+12	−0.684E+10	0.138E+00	0.103E+00	0.794E−01
0.111E+13	0.881E+10	−0.365E+12	0.671E−01	0.107E+00	0.626E−01

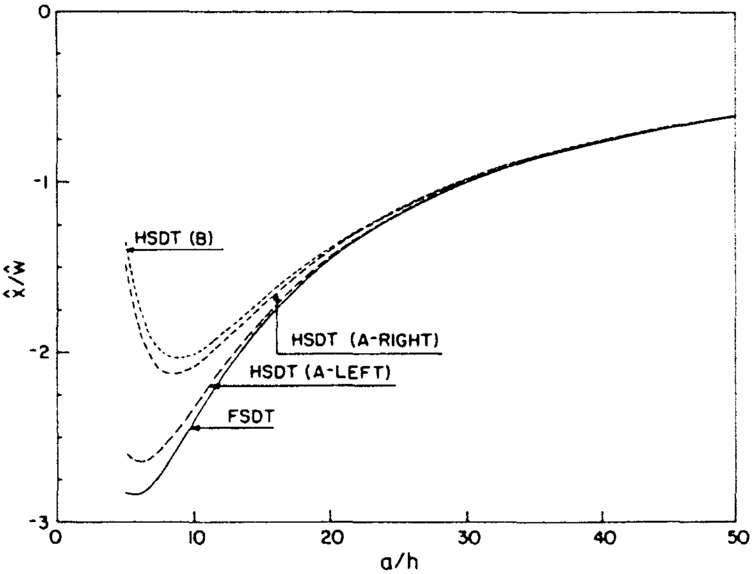

Fig. 3. Variation of $\hat{X}/\hat{W}$ vs a/h (first mode).

displays the normalized fundamental frequencies for various a/h ratios, computed via the first order shear deformation theory, via HSDT (B), and via HSDT (A). All three curves tend asymptotically to the CPT line, drawing closer together as a/h increases. It can also be seen that the curve associated with A is bordered by the two curves associated with B. Figure 3 shows the variation of $\hat{X}$, normalized to $\hat{W}$, for FSDT, HSDT (B) and the right and left of HSDT (A). It can be seen that the right coefficient of HSDT (A) is close to HSDT (B) while the left one is closer to FSDT. Figure 4 shows the variation of $\hat{Y}$, where the curves due to all theories practically coincide. Since shear-deformation theories are efficient at low ratio of a/h, the difference between them is clearly seen there.

The mean-square transverse displacement for the above random vibration problem via the various theories, normalized to those obtained via CPT, is shown in Fig. 5. Also included in this figure are the approximate results obtained by HSDT (A) by using the right eigenvector only. It can be seen that all theories again tend together asymptotically to the

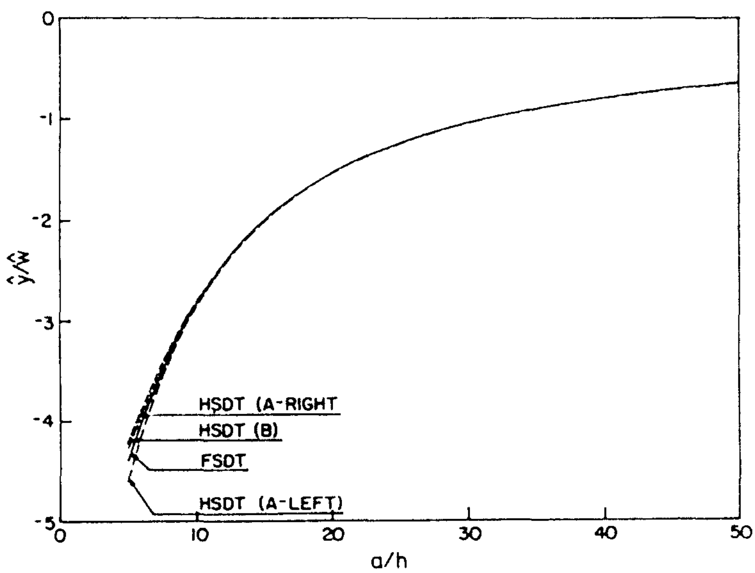

Fig. 4. Variation of $\hat{Y}/\hat{W}$ vs a/h (first mode).

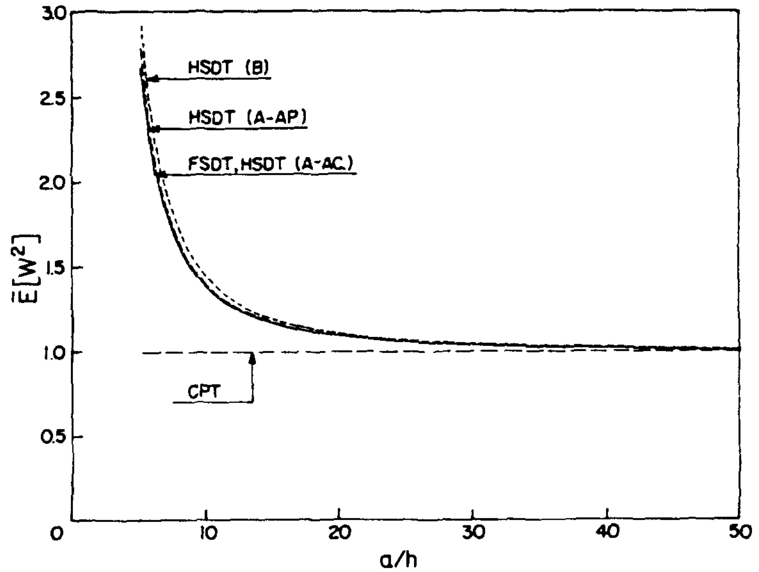

Fig. 5. Normalized (to CPT) mean-square transverse displacement of the plate center vs a/h.

CPT line as the ratio a/h increases, and as in Figs 2–4, the mean-square via HSDT (A) is bordered by those obtained via B.

6. CONCLUSION

The equations governing the dynamics of shear-deformable composite plates are analyzed. This high-order shear deformation theory results in a nonsymmetric operator. Using the first-order perturbation technique, it is shown that the eigenvalues are real and positive, which leads to the conclusion that the operator is positive–definite. The dynamic response of the laminated plate is then formulated using the biorthogonality condition. It was found, that when the right eigenvectors only are used, the results are very close to those obtained when using both the right and the left eigenvectors ones.

The above conclusions could be extended to other plate and shell theories belonging to category A (Ambartsumyan, 1970; Reissner, 1977; Levinson, 1980 and Morley, 1959).

Acknowledgements—G. Cederbaum gratefully appreciates the financial assistance provided by the U.S. Army Missile Commands grant of Professor R. A. Heller, of Virginia Polytechnic Institute and State University. He is also indebted to Professor J. Aboudi of Tel-Aviv University and to Dr J. Ben-Asher of the Virginia Polytechnic Institute and State University for helpful discussions. Partial support of the research reported here by the NASA-Langley Research Center through Grant NAG-1-749 is gratefully acknowledged by L. Librescu. Support by the Research Foundation of the Naval Postgraduate School, Monterey, CA and the Fund for Promotion of Research at the Technion I.I.T. for I. Elishakoff, in the course of this study, is sincerely acknowledged.

REFERENCES

Ambartsumyan, S. A. (1970). *Theory of Anisotropic Plates* (Edited by J. E. Ashton). Technomic, Stanford.

Bert, C. W. and Francis, P. H. (1974). Composite material mechanics: structural mechanics. *AIAA Jnl*, **12**, 1173–1186.

Cederbaum, G., Librescu, L. and Elishakoff, I. (1987). Dynamic response of flat panels made of advanced composite materials subjected to random excitation. 20th Midwestern Mechanics Conference, West Lafayette, Indiana.

Elishakoff, I. (1983). *Probabilistic Methods in the Theory of Structures*. Wiley Interscience, New York.

Franklin, J. N. (1969). *Matrix Theory*. Prentice Hall, Englewood Cliffs, NJ.

Levinson, M. (1980). An accurate, simple theory of the statics and dynamics of elastic plates. *Mech. Res. Commun.* 7, 343–350.

Librescu, L. (1968). Sur les equations de la theorie lineaire des plaques elastiques anisotropes. *C.R. Acad. Sc. Paris*, 443–446.

Librescu, L. and Reddy, J. N. (1987). A critical review and generalization of transverse shear deformable anisotropic plate theories. In *Refined Dynamical Theories of Beams, Plates and Shells and Their Applications* (Edited by I. Elishakoff and H. Irretier), pp. 32–43. Springer, Berlin.

Meirovitch, L. (1980). *Computational Methods in Structural Dynamics*. Sijthoff and Noordhoff, Alphen aan den Rijn, The Netherlands.

Morley, L. S. D. (1959). An improvement of Donnell's approximation of thin-walled circular cylinders. *Quart. J. Mech. appl. Math.* **8**, 89–99.

Pagano, N. J. and Soni, S. A. (1986). Strength analysis of composite turbine blades. First Conference on Composite Materials, Dayton, Ohio.

Reddy, J. N. and Phan, N. D. (1985). Stability and vibration of isotropic, orthotropic and laminated plates according to a higher-order shear deformation theory. *J. Sound Vibr.* **50**, 157–169.

Reissner, E. (1977). On the theory of bending of elastic plates. *J. Math. Phys.* **23**, 189–191.

Ryland, G. and Meirovitch, L. (1980). Response of vibrating system with perturbed parameters. *J. Guidance Control* **3**, 298–303.

Whitney, J. M. and Pagano, N. J. (1970). Shear deformation in heterogeneous anisotropic plates. *J. appl. Mech.* **37**, 1031–1036.

Yang, P. C., Norris, C. H. and Stavsky, Y. (1965). Elastic wave propagation in heterogeneous plates. *Int. J. Solids Structures* **2**, 665–684.